CALIFORNIA STATE SCIENCE FAIR
2013 PROJECT SUMMARY

Name(s)

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Project Number

S1401

Project Title

Dots and Lines: A Combinatorial Interpretation of the Homotopy Groups of Finite Topologies

Abstract

Objectives/Goals

The homotopy groups of a topological space are commonly studied as a topological invariant which gives information about the space's holes, homotopy type, and connectedness. McCord showed the the homotopy groups are precisely those of corresponding simplicial complexes, but gives no explicit way of constructing or interpreting elements of the homotopy groups of a finite space, making applications difficult. An attempt to construct an analogue of homotopy groups for graphs was examined by Atkin and Smith, but the groups they associate to a graph G do not in general correspond to the homotopy groups of a finite topology whose associated graph is G when such a topology exists.

Methods/Materials

My project presents a new definition of the homotopy groups of a graph G, in the spirit of Atkin by using only combinatorial methods, and shows that these homotopy groups of G are indeed the same as any topology whose associated graph is G. In particular, this implies that two topologies with the same associated graph have isomorphic homotopy groups. This new definition allows for a construction with no known analogue in topological homotopy theory, as a graph which reduces the dimension of each homotopy group. More explicitly, for a graph G, I construct a graph G^k whose nth homotopy groupis the same as the (n+k)th homotopy group of G, for all nonnegative integers n. This also allows for algorithms converging to presentations for any homotopy group of any graph (and any simplicial complex after using McCord's correspondence.)

Conclusions/Discussion

The question of the existence of solutions to a certain class of communications routing-type problems is also considered, and using homotopy groups I provide a method of phrasing such problems in terms of groups, which generally simplifies the solution. In fact, in certain cases the existence of a solution can be determined within O(v^3) time (where v is the number of vertices). These problems occur in a variety of situations including chip and network design.

Summary Statement

This project provides a method of constructing all continuous functions (up to pointed homotopy equivalence) from n-spheres to finite topological spaces, generalizes the method to arbitrary graphs, and considers a variety of applications.

Help Received

Father listened while I practiced presentation, Dr. Kubelka at San Jose State University helped review an earlier version of the work (only pointed out errors in grammar/formatting - did not actually contribute to the research).